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# Multivariable Control System Design Algorithm

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**The theory and associated numerical techniques for the Multiple Input Multiple Output Compensator Improvement Program are presented. This algorithm is applicable to the design of the feedback compensation matrix for linear time-invariant control systems with plants possessing multiple control inputs and multiple outputs. The algorithm discussed in this paper differs from the more common multiple-loop break design methods in that each control loop is individually broken. The design specifications that the Multiple Input Multiple Output Compensator Improvement Program attempts to achieve are single input, single output, open-loop frequency response oriented. The practical utility of the program is illustrated by a launch vehicle example.**

## Introduction

BECAUSE of the complexity of the plants and the control laws, the design of modern control systems has become increasingly complicated. In numerous instances classical design techniques (e.g., root locus and frequency response techniques) have been found unsatisfactory when implemented according to standard procedures (basically graphical and trial and error). In particular, the application of these procedures to the design of control systems in which many parameters are selected so that several design objectives are met is a formidable task and in some instances impossible. In order to alleviate the problems associated with the pragmatic use of classical procedures, several computer-aided design algorithms have been developed;[1-7] among these is the Compensator Improvement Program (CIP).[4,5]

The CIP is a computer-aided design tool developed for use by control engineers. The CIP presented in Refs. 4 and 5 was specifically tailored for aiding in the design of feedback compensation for plants with a single control input and multiple outputs.

This paper presents a new CIP that is applicable to the design of compensation for linear time-invariant control systems of plants possessing multiple control inputs and multiple outputs (MIMO). The design philosophy of the MIMO CIP is to modify, in an iterative manner, the free parameters of a compensation transfer function matrix so that the system satisfies specified frequency response design objectives. The underlying principles of the multivariable CIP algorithm are presented, and the practical utility of the program is illustrated with a launch vehicle example.[8]

## Analysis of the Algorithm

Figure 1 is a schematic representation of a multivariable, linear, time-invariant feedback control system. This multivariable system may be viewd as $n$ coupled feedback systems, one for each element of the input vector. The loop transfer function for the $k$th system is obtained by opening the feedback path at $\alpha_k$, and then determining the response $C_k(s)/R_k(s)$ with all other input $R$'s set to zero. With this view of the multivariable feedback system, the designer is faced with the problem of synthesizing controllers of $n$ interacting systems. Using classical feedback theory a controller may be designed so that the open-loop frequency response satisfies a set of frequency response design objectives. In theory this approach is easily extended to the multivariable case. However, simultaneous designs of the controllers must be made so that the $n$ open-loop frequency responses satisfy $n$ sets of design objectives. The simultaneity of the designs is required because of the implicit functional relationship between the design objectives of the individual systems; e.g., a controller may affect the open-loop frequency response of one system in a desirable manner, while adversely affecting the response of another system.

Using classical frequency response techniques, the design of a control system to satisfy a few objectives can be accomplished with manual calculations. However, as the complexity of the system and the number of design objectives increase the development of the controller requires the aid of a digital computer.

The performance measurements of the aforementioned design problem, i.e., phase margins, gain margins, attenuation margins, etc., are very nonlinear functions of the coefficients of the controllers. Except for trivial cases it is impossible to determine the controllers by analytical means. Iterative procedures that are amenable to digital computation are widely used for solving problems of this type; such a procedure is outlined as follows:

1) Compute the open-loop frequency response of each system and evaluate the design performance criteria of each.

2) Determine which performance measurements do not satisfy the design objectives.

3) With respect to the coefficients of the controllers, calculate the gradient vectors of the performance measurements that do not comply with the design objectives.

4) Using these gradient vectors, compute a coefficient change vector that will assure the possibility of simultaneously improving all unsatisfied performance measurements.

5) Using the change vector, increment the coefficients of the controllers.

6) Return to step 1 and repeat the procedure until all design objectives have been met or until no additional improvement in one or more performance measurements can be made.

This step-by-step procedure is a summary of the algorithm from which the MIMO CIP has been developed. A thorough description and explanation of the complete algorithm is given in the Appendix.

To transform the above procedure into a computer code three mathematical developments are required. These are: 1) a technique for calculating the open-loop frequency responses of MIMO systems, 2) a technique for calculating the gradient vectors, and 3) a technique for calculating the coefficient change vector.

### Calculation of Open-Loop Frequency Responses

In order to derive the equations for calculating the open-loop frequency response of each system when all other loops

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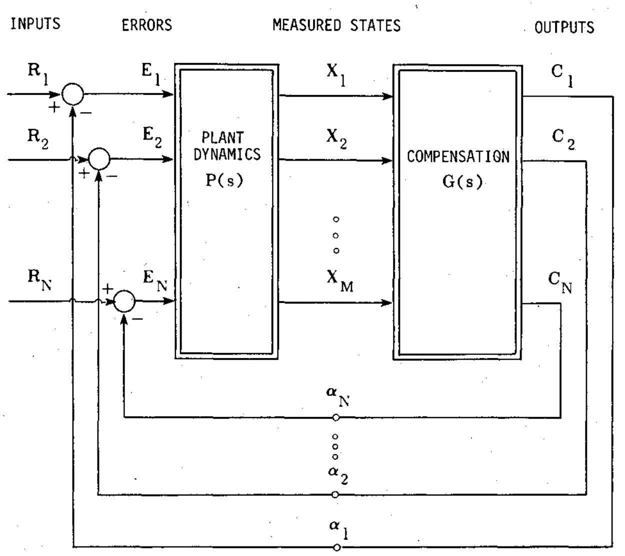
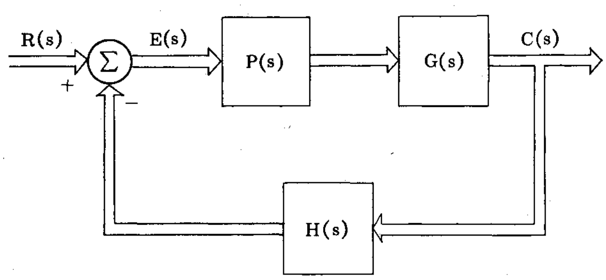



**Fig. 1 Multivariable control system.**




R(s) IS THE N x 1 INPUT VECTOR
E(s) IS THE N x 1 ERROR VECTOR
C(s) IS THE N x 1 OUTPUT VECTOR
P(s) IS THE M x N MATRIX TRANSFER FUNCTION DESCRIBING THE PLANT
G(s) IS THE N x M MATRIX TRANSFER FUNCTION DESCRIBING THE CONTROL LAW AND CASCADED COMPENSATION
H(s) IS AN N x N UNITY MATRIX

**Fig. 2 Vector-matrix representation of a multivariable feedback control system.**

are closed, attention is focused on Fig. 2. This illustration is a matrix representation of Fig. 1 with the addition of the feedback matrix $[H(s)]$. The closed-loop output vector $[C(s)]$ is

$$[C(s)] = [G(s)][P(s)]\{I+[H(s)] \times [G(s)][P(s)]\}^{-1}[R(s)] \tag{1}$$

If $[H(s)]$ is a diagonal matrix except for the $k$th element which is zero and all the elements of $[R(s)]$ are zero except the $k$th element which is unity, the result is the impulse response between the $k$th input and the outputs when the $k$th loop is open. Letting

$$[V(s)] \triangleq \{I+[H(s)][G(s)][P(s)]\}^{-1} \tag{2}$$

and

$$[U(s)] \triangleq [G(s)][P(s)] \tag{3}$$

where $s=j\omega$. It is easily seen that the open-loop frequency response of the $k$th system is

$$C_k(j\omega)/R_k(j\omega) = u^k(j\omega)v_k(j\omega) \tag{4}$$

where $C_k(j\omega)$ and $R_k(j\omega)$ are the $k$th elements of $[C(j\omega)]$ and $[R(j\omega)]$, and $u^k(s)$ and $v_k(s)$ are the $k$th row of $[U(s)]$ and $k$th column of $[V(s)]$, respectively. By setting the proper diagonal element of $[H(s)]$ to zero, Eqs. (2-4) can be used to evaluate the open-loop frequency response of each system.

### Determination of Partial Vectors

In the MIMO CIP the performance measurements are improved by perturbing corresponding frequency response points with respect to judiciously selected points in the corresponding complex plane. Then a typical performance measurement function for the $k$th open-loop system has the form

$$J(w) = \{[A+C_k(j\omega)][A+C_k(j\omega)]^*\}^{1/2} \tag{5}$$

where $A$ is the point in the complex plane from which the perturbation is to occur and $w$ is a parameter that can be varied (transfer function coefficient). For stability and attenuation margins, the values of $A$ are, respectively, the $(-1+j0)$ and $(0+j0)$ points. For perturbation purposes, the $A$ for gain and phase margins assumes floating values that are 5 units along lines perpendicular to the tangent lines at the corresponding critical points on the polar frequency response. However, for objective function evaluations the $A$'s for gain and phase margins are selected as the $(-1+j0)$ point.

The partial derivative of $J(w)$ with respect to $w$ is

$$\frac{\partial J}{\partial w} = \mathrm{Re}\left\{[A+C_k(j\omega)]^* \frac{\partial C_k(j\omega)}{\partial w}\right\} \Big/ J \tag{6}$$

The term $\partial C_k(j\omega)/\partial w$ can be expanded by the chain rule as

$$\frac{\partial C_k(j\omega)}{\partial w} = \frac{\partial C_k(j\omega)}{\partial G_{ij}} \cdot \frac{\partial G_{ij}}{\partial w} \tag{7}$$

where $G_{ij}$ is the $(ij)$th element of the matrix $[G(s)]$ in which the parameter $w$ appears. Evaluation of the first term in Eq. (7) yields

$$\frac{\partial[C(s)]}{\partial G_{ij}} = -[G(s)][P(s)]\{I+[H(s)][G(s)][P(s)]\}^{-1}[H(s)] \times \frac{\partial[G(s)]}{\partial G_{ij}}[P(s)]\{I+[H(s)][G(s)][P(s)]\}^{-1}[R(s)] + \frac{\partial[G(s)]}{\partial G_{ij}}[P(s)]\{I+[H(s)][G(s)][P(s)]\}^{-1}[R(s)] \tag{8}$$

In Eq. (8), $\partial[G(s)]/\partial G_{ij}$ is a zero matrix except for the $(ij)$th element which is unity. Then $\partial C_k(s)/\partial G_{ij}$ is the $k$th element of Eq. (8) with the $k$th diagonal element of $[H(s)]$ set to zero, and all the elements of $[R(s)]$ set to zero except the $k$th which is set to unity.

Evaluation of the second term in Eq. (7) is accomplished by assuming that the $(ij)$th element of the compensator matrix $[G(s)]$ is composed of a cascaded arrangement of transfer functions, i.e.,

$$G_{ij}(s) = \prod_{k=1}^{K} G_{ijk}(s) \tag{9}$$

**Table 1 Plant dynamics describing the yaw/roll ascent flight control for the space shuttle**

| Data Point | Frequency RE[s] / IM[s] | Uncompensated Plant Frequency Response Data: RE[$P_{11}$] / IM[$P_{11}$] | RE[$P_{12}$] / IM[$P_{12}$] | RE[$P_{21}$] / IM[$P_{21}$] | RE[$P_{22}$] / IM[$P_{22}$] | RE[$P_{31}$] / IM[$P_{31}$] | RE[$P_{32}$] / IM[$P_{32}$] | RE[$P_{41}$] / IM[$P_{41}$] | RE[$P_{42}$] / IM[$P_{42}$] |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0000 | -2.8340 | -0.0431 | -10.6900 | 1.0700 | 2.7380 | 0.1860 | 31.2900 | -5.8820 |
| | 0.0100 | -3.2070 | -0.0463 | -2.3320 | -0.1385 | -13.7400 | 2.1720 | 22.7100 | -2.7820 |
| 2 | 0.0000 | -0.7722 | -0.0117 | -2.6520 | 1.1850 | -0.1172 | 0.1527 | 3.6560 | -1.9500 |
| | 0.0185 | -1.1690 | -0.0171 | -0.1011 | -0.2079 | -2.6890 | 1.2350 | 1.9790 | -1.6310 |
| 3 | 0.0000 | -0.6223 | -0.0095 | -2.1060 | -1.1910 | -0.1571 | 0.1525 | 2.5500 | -1.5740 |
| | 0.0209 | -1.0030 | -0.0147 | -0.1604 | -0.2360 | -2.0880 | 1.0930 | 1.3180 | -1.4430 |
| 4 | 0.0000 | -0.4073 | -0.0063 | -1.3360 | 1.1950 | -0.1815 | 0.1522 | 1.2320 | -1.0060 |
| | 0.0275 | -0.7280 | -0.0108 | 0.2186 | -0.3130 | -1.2750 | 0.8303 | 0.5690 | -1.0950 |
| 5 | 0.0000 | -0.1988 | -0.0036 | -0.6279 | 1.1670 | -0.1668 | 0.1486 | 0.3088 | -0.4414 |
| | 0.0507 | - -.3513 | -0.0059 | 0.2656 | -0.5978 | -0.5097 | 0.4312 | 0.1152 | -0.5680 |
| 6 | 0.0000 | -0.1948 | -0.0045 | -0.5711 | 1.1680 | -0.1802 | 0.1479 | 0.3197 | -0.4330 |
| | 0.0517 | -0.3422 | -0.0059 | -0.0029 | -0.6239 | -0.4985 | 0.4219 | 0.1078 | -0.5562 |
| 7 | 0.0000 | -0.1940 | -0.0049 | -0.8675 | 1.1520 | -0.1878 | 0.1476 | 0.3208 | -0.4318 |
| | 0.0519 | -0.3403 | -0.0049 | -0.1351 | -0.6331 | -0.4795 | 0.4205 | 0.0795 | -0.5532 |
| 8 | 0.0000 | -0.6462 | -0.2679 | -1.2820 | 1.1400 | 0.6761 | 0.1210 | -0.0047 | -0.3364 |
| | 0.0659 | -0.1136 | -0.0085 | 5.0450 | -0.9269 | -0.0904 | 0.3103 | -0.3594 | -0.4264 |
| 9 | 0.0000 | 0.5705 | -0.0246 | 4.4650 | 0.9682 | 0.7388 | 0.1190 | 0.4012 | -0.3467 |
| | 0.0662 | -1.1130 | 0.0212 | 4.4600 | -0.9142 | -1.1540 | 0.3393 | 0.4899 | -0.4268 |
| 10 | 0.0000 | -0.1037 | -0.0046 | 3.7080 | 0.9895 | 0.0113 | 0.1401 | 0.4222 | -0.3450 |
| | 0.0665 | -0.9970 | 0.0179 | 0.5986 | -0.8032 | -1.1280 | 0.3372 | 0.1397 | -0.4138 |
| 11 | 0.0000 | -0.2463 | -0.0002 | 0.4308 | 1.0820 | -0.2220 | 0.1466 | 0.2557 | -0.3347 |
| | 0.0679 | -0.4227 | 0.0012 | -0.1450 | -0.7998 | -0.5402 | 0.3123 | 0.0532 | -0.4019 |
| 12 | 0.0000 | -0.1350 | -0.0016 | -0.3108 | 0.7750 | -0.1569 | 0.1307 | 0.0841 | -0.2253 |
| | 0.1181 | -0.1243 | -0.0016 | 0.6162 | -1.5550 | -0.1698 | 0.1414 | 0.0189 | -0.1819 |
| 13 | 0.0000 | -0.1170 | -0.0015 | 0.0244 | 0.0049 | -0.1471 | 0.1175 | 0.0641 | -0.1865 |
| | 0.1639 | -0.0639 | -0.0009 | 1.1130 | -2.3520 | -0.0906 | 0.0737 | 0.0071 | -0.0914 |
| 14 | 0.0000 | -0.1046 | -0.0016 | 1.4280 | -2.3960 | -0.1408 | 0.1040 | 0.0581 | -0.1579 |
| | 0.2246 | -0.0189 | -0.0004 | 1.6570 | -3.0610 | -0.0301 | 0.0240 | -0.0057 | -0.0239 |
| 15 | 0.0000 | -0.1323 | -0.0066 | 7.6720 | -11.9200 | -0.1912 | 0.1158 | 0.0816 | -0.1780 |
| | 0.3068 | 0.0349 | 0.0019 | -0.6730 | 0.5485 | 0.0449 | -0.0246 | -0.0363 | 0.0485 |
| 16 | 0.0000 | -0.5012 | -0.1355 | 37.5100 | -76.2100 | -0.7038 | 0.3409 | 0.2381 | -0.7026 |
| | 0.3412 | 0.4944 | 0.3420 | -18.1200 | 78.4600 | 0.5483 | -0.1452 | -0.1522 | 0.7367 |
| 17 | 0.0000 | -0.2754 | 0.6667 | 90.5100 | -7.2760 | -0.8031 | 0.6909 | 0.6666 | -4.2170 |
| | 0.3434 | 0.9189 | 0.7194 | -29.0500 | 168.8000 | 0.9950 | -0.2849 | -0.1838 | 1.5530 |
| 18 | 0.0000 | 0.7809 | 0.0304 | -89.6700 | 120.3000 | 1.1130 | -0.5521 | -0.7124 | 1.0020 |
| | 0.3469 | 1.3760 | -0.8157 | -220.4000 | 137.1000 | 2.5630 | -1.5400 | -1.6360 | 1.0260 |
| 19 | 0.0000 | -0.0097 | 0.0065 | -3.1110 | 2.2820 | -0.0285 | 0.0242 | 0.0258 | -2.4050 |
| | 0.4312 | 0.0115 | -0.0080 | 5.8500 | -4.6220 | 0.0254 | -0.0282 | 0.0224 | 0.0314 |
| 20 | 0.0000 | 0.3911 | -0.1390 | 63.6500 | -22.3600 | 1.1070 | -0.3858 | -1.0980 | 0.3797 |
| | 0.4917 | 0.2532 | -0.0870 | 10.1100 | -4.3860 | 0.7313 | -0.2731 | -0.6346 | 0.2498 |
| 21 | 0.0000 | 0.0027 | -0.0026 | 0.9412 | -0.6377 | 0.0001 | -0.0066 | 0.0044 | 0.0151 |
| | 0.5654 | 0.0027 | -0.0019 | 1.0500 | -0.9433 | -0.0011 | -0.0299 | -0.0375 | 0.0538 |
| 22 | 0.0000 | 0.2691 | -0.0726 | 18.2600 | -5.4870 | 1.2830 | -0.3560 | -1.4590 | 0.4108 |
| | 0.6600 | -0.1539 | 0.0343 | -41.9300 | 9.9780 | -0.7248 | 0.1372 | 0.8950 | -0.1667 |
| 23 | 0.0000 | -0.0263 | -0.0005 | -17.4500 | 3.5250 | -0.1362 | -0.0131 | -0.3378 | 0.1425 |
| | 0.6677 | -0.1667 | 0.0404 | -16.6400 | 4.1580 | -0.8415 | 0.1809 | 1.4090 | -0.3044 |

where $K$ represents the number of cascaded elements. The $l$th element of the $(ij)$th compensator has the general form

$$G_{ijl}(s) = \frac{\sum_{m=0}^{M} x_{ijlm} s^m}{\sum_{n=0}^{N} y_{ijln} s^n} \tag{10}$$

where the free parameters of this element are the $x$'s and $y$'s§.

Assuming the parameter $w$ in Eq. (7) is the $p$th numerator coefficient in Eq. (10), then the following expression is obtained:

$$\frac{\partial G_{ij}(s)}{\partial x_{ijlp}} = G_{ij}(s) \frac{+s^p}{\sum_{m=0}^{M} x_{ijlm} s^m} \tag{11}$$

Similarly, letting $w$ represent the $p$th denominator coefficient in Eq. (7), then

$$\frac{\partial G_{ij}(s)}{\partial y_{ijlp}} = G_{ij}(s) \frac{-s^p}{\sum_{n=0}^{N} y_{ijln} s^n} \tag{12}$$

Thus, Eqs. (8), (11), and (12) can be used effectively to determine the first order change of any CIP performance measurement function with respect to the free parameters of the controller.

#### Computation of the Directional Vector

The constraint improvement technique (CIT) is used to compute the directional vector $\boldsymbol{d}$. The basis of this algorithm is delineated in the following paragraphs. The directional vector is computed as a weighted sum of the gradients of the active constraints (those performance measurements not complying with the design objectives); that is,

$$\boldsymbol{d} = [\nabla G]\boldsymbol{a} \tag{13}$$

§Actually in the MIMO CIP, $M$ and $N$ are only allowed to be 1 or 2.

**Table 2 Initial compensation matrix**

| | | | | | |
|---|---|---|---|---|---|
| COMPENSATOR (1,1): GAIN = 1.00000 | | | | | |
| COMPENSATOR COEFFICIENTS | | | | | |
| $ZA_1$ = | .100000-01 | $ZB_1$ = | 1.00000 | | |
| $ZC_1$ = | 1.00000 | $ZD_1$ = | 1.00000 | $ZE_1$ = | 8.16300 |
| $ZC_2$ = | 1.00000 | $ZD_2$ = | 2.28600 | $ZE_2$ = | 2.01400 |
| $ZC_3$ = | 1.00000 | $ZD_3$ = | 1.20000 | $ZE_3$ = | 4.44400 |
| $ZC_4$ = | 1.00000 | $ZD_4$ = | 1.00000 | $ZE_4$ = | 8.16300 |
| $PA_1$ = | .133000-01 | $PB_1$ = | 1.00000 | | |
| $PC_1$ = | 1.00000 | $PD_1$ = | 4.34800 | $PE_1$ = | 18.90300 |
| $PC_2$ = | 1.00000 | $PD_2$ = | 2.34800 | $PE_2$ = | 18.90300 |
| $PC_3$ = | 1.00000 | $PD_3$ = | 2.40000 | $PE_3$ = | 4.00000 |
| $PC_4$ = | 1.00000 | $PD_4$ = | 4.34800 | $PE_4$ = | 18.90300 |
| COMPENSATOR (1,2): GAIN = .74500 | | | | | |
| COMPENSATOR COEFFICIENTS: | | | | | |
| $ZA_1$ = | 1.00000 | $ZB_1$ = | -1.00000 | | |
| $ZA_1$ = | .100000-01 | $ZB_1$ = | 1.00000 | | |
| $ZC_1$ = | 1.00000 | $ZD_1$ = | .571400 | $ZE_1$ = | 8.16300 |
| $ZC_2$ = | 1.00000 | $ZD_2$ = | .645200 | $ZE_2$ = | 10.41000 |
| $ZC_3$ = | 1.00000 | $ZD_3$ = | .571400 | $ZE_3$ = | 2.04100 |
| $ZC_4$ = | 1.00000 | $ZD_4$ = | 1.200000 | $ZE_4$ = | 4.44400 |
| $PA_1$ = | .133000-01 | $PB_1$ = | 50.00740 | | |
| $PA_1$ = | 1.00000 | $PB_1$ = | 1.00000 | | |
| $PC_1$ = | 1.00000 | $PD_1$ = | 5.21700 | $PE_1$ = | 18.9030 |
| $PC_2$ = | 1.00000 | $PD_2$ = | 6.00000 | $PE_2$ = | 25.0000 |
| $PC_3$ = | 1.00000 | $PD_3$ = | 5.33300 | $PE_3$ = | 11.1111 |
| $PC_4$ = | 1.00000 | $PD_4$ = | 2.40000 | $PE_4$ = | 4.00000 |
| COMPENSATOR (1,3): GAIN = .74500 | | | | | |
| COMPENSATOR COEFFICIENTS | | | | | |
| $ZA_1$ = | .00000 | $ZB_1$ = | 51.0074 | | |
| $ZA_1$ = | .100000-01 | $ZB_1$ = | 1.00000 | | |
| $ZC_1$ = | 1.00000 | $ZD_1$ = | .571400 | $ZE_1$ = | 8.16300 |
| $ZC_2$ = | 1.00000 | $ZD_2$ = | .645200 | $ZE_2$ = | 10.4100 |
| $ZC_3$ = | 1.00000 | $ZD_3$ = | .571400 | $ZE_3$ = | 2.04100 |
| $ZC_4$ = | 1.00000 | $ZD_4$ = | 1.200000 | $ZE_4$ = | 4.44400 |
| $PA_1$ = | 1.00000 | $PB_1$ = | 50.0074 | | |
| $PA_1$ = | .133000-01 | $PB_1$ = | 1.00000 | | |
| $PC_1$ = | 1.00000 | $PD_1$ = | 5.21700 | $PE_1$ = | 18.9030 |
| $PC_2$ = | 1.00000 | $PD_2$ = | 6.00000 | $PE_2$ = | 25.0000 |
| $PC_3$ = | 1.00000 | $PD_3$ = | 5.33300 | $PE_3$ = | 11.1110 |
| $PC_4$ = | 1.00000 | $PD_4$ = | 2.40000 | $PE_4$ = | 4.00000 |
| COMPENSATOR (2,4): GAIN = .74500 | | | | | |
| COMPENSATOR COEFFICIENTS | | | | | |
| $ZA_1$ = | .300000-01 | $ZB_1$ = | 1.00000 | | |
| $ZC_1$ = | 1.00000 | $ZD_1$ = | .857000 | $ZE_1$ = | 8.16300 |
| $ZC_2$ = | 1.00000 | $ZD_2$ = | 1.00000 | $ZE_2$ = | 6.25000 |
| $ZC_3$ = | 1.00000 | $ZD_3$ = | 2.00000 | $ZE_3$ = | 1.56300 |
| $ZC_4$ = | 1.00000 | $ZD_4$ = | 1.20000 | $ZE_4$ = | 4.44400 |
| $PA_1$ = | .40000-01 | $PB_1$ = | 1.00000 | | |
| $PC_1$ = | 1.00000 | $PD_1$ = | 4.34800 | $PE_1$ = | 18.9030 |
| $PC_2$ = | 1.00000 | $PD_2$ = | 4.34800 | $PE_2$ = | 18.9030 |
| $PC_3$ = | 1.00000 | $PD_3$ = | 3.33300 | $PE_3$ = | 11.1110 |
| $PC_4$ = | 1.00000 | $PD_4$ = | 2.40000 | $PE_4$ = | 4.00000 |
| COMPENSATORS HAVING ZERO CONTRIBUTION: (1,4), (2,1), (2,2), (2.3) | | | | | |

where $[\nabla G]$ is a matrix whose columns are the gradients of the active constraints and the vector $a$ contains the weighting constants. It is desirable that the components of $a$ be selected so that it is feasible to improve simultaneously each active constraint. Such a feasibility exists if the inner product of the directional vector with each column of $[\nabla G]$ is a positive number.¶ Then the optimal values of the weighting constants are

$$a = \{[\nabla G]^T[\nabla G]\}^{-1}c \tag{14}$$

where $[\ ]^T$ means transpose and $c$ is a vector containing the desired inner products between $d$ and the columns of $[\nabla G]$.

By controlling the values of the elements of $c$ certain active constraints can be weighed more heavily in the determination of the directional vector while still maintaining the feasibility of improving each active constraint. However, in implementing the CIT in the MIMO CIP it has been found convenient to normalize the columns of $[\nabla G]$ to unit vectors and thus select the elements of $c$ as unity.

## Launch Vehicle Example

This example is representative of the yaw/roll ascent flight control system for the Space Shuttle. The system is similar to that of Fig. 1 with a plant possessing two control inputs and four outputs. The 23 frequency response data points chosen to describe the plant dynamics are listed in Table 1. The coefficients of the elements of the compensation matrix $[G(s)]$ are given in Table 2. This controller represents a proposed design for use on the Space Shuttle; however, all design objectives are not met. Part A of Table 3 presents the margin values, the frequency points where they occur, the margin type, and denotes whether a particular margin is active. For both systems three are a total of three margins unsatisfied. The MIMO CIP has been used to improve these unsatisfied margins while requiring that the dc gains of the compensators remain unchanged, thus, not affecting the steady-state error criterion. After five iterations (20 s of CPU time on a Univac 1106) all system design specifications are obtained. The margin values are shown in part B of Table 3 and the compensator coefficients are shown in Table 4.

¶In order to simplify the explanation of the CIT it is implicitly assumed that an active constraint is improved if its value is increased.

**Table 3 System specifications**

| Margin Number | Margin Value | Complex Frequency RE[s] | Complex Frequency IM[s] | Desired Margin | Margin Type | Active List |
|---|---|---|---|---|---|---|
| A. Subsystem No. 1, Iteration No. 0 | | | | | | |
| 1 | 0.4163 | 0.000 | 0.0896 | 0.60 | GM | Yes |
| 2 | 53.7400 | 0.000 | 0.0240 | 30.00 | PM | No |
| 3 | 127.5000 | 0.000 | 0.0603 | 30.00 | PM | No |
| 4 | 24.2100 | 0.000 | 0.0679 | 30.00 | PM | Yes |
| 5 | 0.0074 | 0.000 | 0.3412 | 0.10 | AM | No |
| 6 | 0.0289 | 0.000 | 0.3469 | 0.10 | AM | No |
| 7 | 0.0097 | 0.000 | 0.6600 | 0.10 | AM | No |
| Subsystem No. 2, Iteration No. 0 | | | | | | |
| 1 | 0.5941 | 0.000 | 0.0896 | 0.60 | GM | Yes |
| 2 | 36.5000 | 0.000 | 0.0321 | 30.00 | PM | No |
| 3 | 0.0885 | 0.000 | 0.3434 | 0.10 | AM | No |
| 4 | 0.0076 | 0.000 | 0.6600 | 0.10 | AM | No |
| B. The Improved System Specifications Subsystem No. 1, Iteration No.5 | | | | | | |
| 1 | 0.6120 | 0.000 | 0.0927 | 0.60 | GM | No |
| 2 | 38.1800 | 0.000 | 0.0222 | 30.00 | PM | No |
| 3 | 143.9000 | 0.000 | 0.0616 | 30.00 | PM | No |
| 4 | 32.0100 | 0.000 | 0.0674 | 30.00 | PM | No |
| 5 | 0.0307 | 0.000 | 0.3469 | 0.10 | AM | No |
| 6 | 0.0098 | 0.000 | 0.4917 | 0.10 | AM | No |
| 7 | 0.0041 | 0.000 | 0.6600 | 0.10 | AM | No |
| Subsystem No. 2, Iteration No. 5 | | | | | | |
| 1 | 0.6075 | 0.000 | 0.0896 | 0.60 | GM | No |
| 2 | 31.3600 | 0.000 | 0.0339 | 30.00 | PM | No |
| 3 | 0.0845 | 0.000 | 0.03434 | 0.10 | AM | No |
| 4 | 0.0076 | 0.000 | 0.6600 | 0.10 | AM | No |

Note: GM is gain margin; PM is phase margin; and AM is attenuation margin.

## Conclusions

Because of the complexity of technology and control laws, the design of modern control systems has become increasingly complicated. In this paper, the theory and associated numerical techniques for achieving a computer-aided compensation design improvement algorithm (MIMO CIP) for multivariable control systems have been presented. The technique developed is applicable to the design of compensation for linear time-invariant plants possessing multiple control inputs and multiple outputs. The underlying principle of the MIMO CIP is to modify in an iterative manner the free parameters of the compensation so that the system satisfies specified frequency response properties. The algorithm has been implemented in Fortran IV and is available to the public.

In conclusion, the work presented in this paper demonstrates that the classical control theory is amenable to systems with multivariable characteristics. An important benefit derived from the use of the frequency domain for linear time-invariant systems is the intuition it provides in determining the soundness of the system.

**Table 4 Final compensation matrix**

| | | |
|---|---|---|
| COMPENSATOR (1,1): GAIN = 1.00000 | | |
| COMPENSATOR COEFFICIENTS: | | |
| $ZA_1$ = .100000-01 | $ZB_1$ = .744312 | |
| $ZC_1$ = 1.00000 | $ZD_1$ = 1.02373 | $ZE_1$ = 8.16526 |
| $ZC_2$ = 1.00000 | $ZD_2$ = 2.30558 | $ZE_2$ = 2.01630 |
| $ZC_3$ = 1.00000 | $ZD_3$ = 1.22262 | $ZE_3$ = 4.44622 |
| $ZC_4$ = 1.00000 | $ZD_4$ = 1.02373 | $ZE_4$ = 8.16526 |
| $PA_1$ = .133000-01 | $PB_1$ = 1.18973 | |
| $PC_1$ = 1.00000 | $PD_1$ = 4.36098 | $PE_1$ = 18.9002 |
| $PC_2$ = 1.00000 | $PD_2$ = 4.36098 | $PE_2$ = 18.9002 |
| $PC_3$ = 1.00000 | $PD_3$ = 2.38040 | $PE_3$ = 3.99765 |
| $PC_4$ = 1.00000 | $PD_4$ = 4.36098 | $PE_4$ = 18.9002 |
| COMPENSATOR (1,2): GAIN = .74500 | | |
| COMPENSATOR COEFFICIENTS | | |
| $ZA_1$ = 1.00000 | $ZB_1$ = -.996201 | |
| $ZA_2$ = .100000-01 | $ZB_2$ = 1.01163 | |
| $ZC_1$ = 1.00000 | $ZD_1$ = .582803 | $ZE_1$ = 8.16205 |
| $ZC_2$ = 1.00000 | $ZD_2$ = .656982 | $ZE_2$ = 10.4091 |
| $ZC_3$ = 1.00000 | $ZD_3$ = .576605 | $ZE_3$ = 2.04025 |
| $ZC_4$ = 1.00000 | $ZD_4$ = 1.20585 | $ZE_4$ = 4.44328 |
| $PA_1$ = 1.00000 | $PB_1$ = 50.0069 | |
| $PA_2$ = .133300-01 | $PB_2$ = .983323 | |
| $PC_1$ = 1.00000 | $PD_1$ = 5.20890 | $PE_1$ = 18.9033 |
| $PC_2$ = 1.00000 | $PD_2$ = 5.99161 | $PE_2$ = 25.0002 |
| $PC_2$ = 1.00000 | $PD_3$ = 5.32565 | $PE_3$ = 11.1114 |
| $PC_4$ = 1.00000 | $PD_4$ = 2.39359 | $PE_4$ = 4.00060 |
| COMPENSATOR (1,3): GAIN = .74500 | | |
| COMPENSATOR COEFFICIENTS | | |
| $ZA_1$ = .000000 | $ZB_1$ = 51.0048 | |
| $ZA_2$ = .100000-01 | $ZB_2$ = .869534 | |
| $ZC_1$ = 1.00000 | $ZC_1$ - .583262 | $ZE_1$ = 8.16477 |
| $ZC_2$ = 1.00000 | $ZD_2$ = .657081 | $ZE_2$ = 10.4118 |
| $ZC_3$ = 1.00000 | $ZD_3$ = .582806 | $ZE_3$ = 2.04267 |
| $ZC_4$ = 1.00000 | $ZD_4$ = 1.21041 | $ZE_4$ = 4.44575 |
| $PA_1$ = 1.00000 | $PB_1$ = 50.0095 | |
| $PA_2$ = .133000-01 | $PB_2$ = 1.10453 | |
| $PC_1$ = 1.00000 | $PD_1$ = 5.21469 | $PE_1$ = 18.9011 |
| $PC_2$ = 1.00000 | $PD_2$ = 5.99959 | $PE_2$ = 24.9981 |
| $PC_3$ = 1.00000 | $PD_3$ = 5.33026 | $PE_3$ = 11.1092 |
| $PC_4$ = 1.00000 | $PD_4$ = 2.39180 | $PE_4$ = 3.99822 |
| COMPENSATOR (2,4): GAIN = 1.0000 | | |
| COMPENSATOR COEFFICIENTS | | |
| $ZA_1$ = .300000-01 | $ZB_1$ = .995009 | |
| $ZC_1$ = 1.00000 | $ZD_1$ = .827221 | $ZE_1$ = 8.16655 |
| $ZC_2$ = 1.00000 | $ZD_2$ = .992354 | $ZE_2$ = 6.25852 |
| $ZC_3$ = 1.00000 | $ZD_3$ = 2.00871 | $ZE_3$ = 1.56803 |
| $ZC_4$ = 1.00000 | $PD_4$ = 1.20369 | $ZE_4$ = 4.45146 |
| $PA_1$ = .400000-01 | $PB_1$ = 1.00117 | |
| $PC_1$ = 1.00000 | $PD_1$ = 4.34837 | $PE_1$ = 18.9029 |
| $PC_2$ = 1.00000 | $PD_2$ = 4.34837 | $PE_2$ = 18.9029 |
| $PC_3$ = 1.00000 | $PD_3$ = 3.33705 | $PE_3$ = 11.1106 |
| $PC_4$ = 1.00000 | $PD_4$ = 2.39557 | $PE_4$ = 3.99576 |
| COMPENSATORS HAVING ZERO CONTRIBUTION: (1,4), (2,1), (2,2), (2,3) | | |

## Appendix: Description of the Design Algorithm

The algorithm from which the MIMO CIP has been developed is shown in Fig. 3. It can be broken into two major parts, the data input and the iterative loop. A description of these parts follows.

### Data Input

The first part of the algorithm is the input of the necessary data for initialization. Table 5 summarizes the data required for the MIMO CIP. The input data is described in four categories.

First, values for iteration control parameters are required. Included in these are extremes on step size to be taken on iterations, maximum iterations per execution, designation of iterations to be printed, user identification code, etc. The user must also specify the mode used in the program to determine when an iteration has been completed; the mode designates which continuance criterion is to be used for determining whether the trial design at the $(i+1)$th iteration is an improvement in comparison to the results at the $i$th iteration. One of the following two modes must be selected:

1) Total improved frequency response mode (TIFR) requires that from iteration to iteration no unsatisfied objectives or design specifications are allowed to degrade and insures improvement in at least one.

**Table 5 Outline of CIP data**

1) Iteration control
   a) Mode, identification code
   b) Start, stop, print iterations
   c) Maximum, minimum step sizes, etc.

2) Design specifications
   a) Desired stability and attenuation radii
   b) Frequency ranges over which searches for critical points are to be made

3) Description of plant
   a) Numer of control inputs
   b) Number of outputs
   c) Discrete frequency response data

4) Description of compensation
   a) Gain constant in each channel
   b) Number of subcompensators in each channel
   c) Coefficients for each subcompensator in first- and second-order factors only
   d) Constraints to be placed on the coefficients

2) Sum improved frequency response mode (SIFR) requires that the sum improvement exceeds the sum degradation from iteration to iteration.

It is obvious that the TIFR mode produces a more stringent continuance criterion.

The second category of the input data designates the design specifications. There are four types of specifications that must be made: gain margins, phase margins, stability margins, and attenuation margins. The first three margins are used to define desired phase stabilization, and the last defines the desired gain stabilization. The frequency dependence of each margin must be defined; additionally, for each margin the user specifies a frequency range on which to search for these critical points.

The third category of the input data is the description of the plant. The uncompensated plant is described by the frequency response data between each input and output channel for a given set of frequencies. The choice of the discrete data description for the plant was made to avoid computational difficulties that might be encountered in evaluating high-order transfer functions and to conserve computing time in the iterative process of the CIP.

The fourth category given in Table 5 is a description of the elements of the compensation matrix $[G(s)]$. In this regard, the designer must select the control law necessary to achieve the design objectives. Designs for continuous systems are performed in the $s$-domain, whereas designs for sampled-data systems are performed in the $w$-domain. Thus for simplicity the form of the elements of the compensation matrix are transfer functions in the form of cascade first- and second-order factors. For example in the $s$-domain, the $(kl)$ element has the general form

$$G_{kl}(s) = (\text{gain}) \frac{\prod_{i=1}^{N1} (ZA_i + ZB_i s) \prod_{j=1}^{N2} (Z_{C_j} + ZD_j s + ZE_j s^2)}{\prod_{i=1}^{M1} (PA_i + PB_i s) \prod_{j=2}^{M2} (PC_j + PD_j s + PE_j s^2)} \tag{A1}$$

For each element of the compensator matrix the user must specify initial values of gain, $N1$, $N2$, $M1$, $M2$, $ZA$'s, $ZB$'s, $ZC$'s, $ZD$'s, $ZE$'s, $PA$'s, etc; it may be specified that the dc terms remain unaltered.

In addition allowances have been made for constraining particular compensator coefficients. For first-order

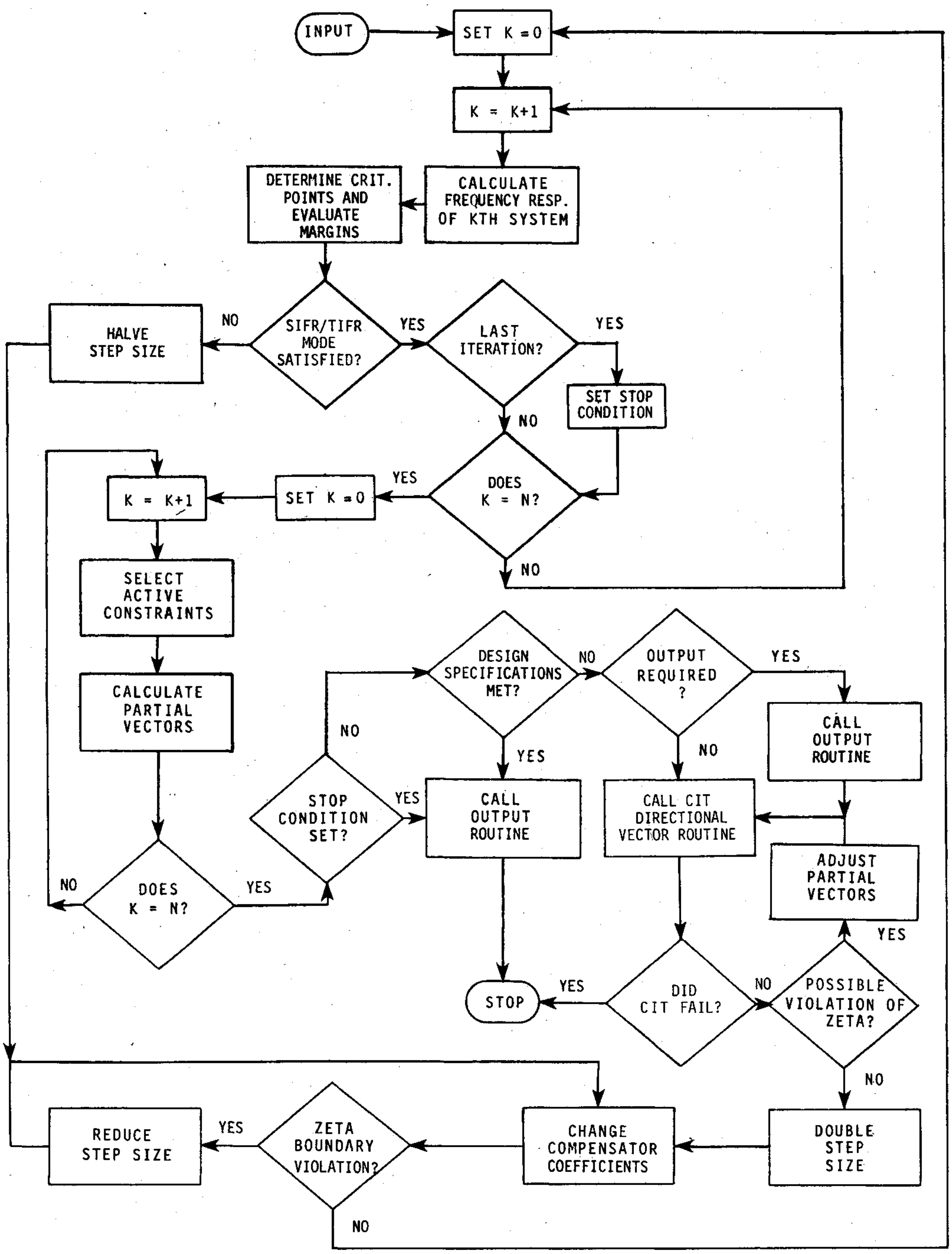



**Fig. 3 Simplified flow diagram for the MIMO CIP algorithm.**

numerator and/or denominator factors the user can specify whether negative coefficients are allowable; for second-order factor zeta boundaries may be defined which prohibit the roots of these factors from violating a specified damping ratio sector.

### Iterative Loop

The second portion of the algorithm is the actual iterative design loop; as evident in Fig. 3 this begins immediately after the input block. The first two steps of this loop involve the setting and incrementing of the system counter $K$ in correspondence with the number of plant inputs, hence the number of the subsystem involved. Next the frequency response of the $K$th system is calculated by breaking a single loop at $\alpha_k$; the respective critical points are determined along with the corresponding values of the margins.

The values of the margins are used to determine if the chosen continuance criterion mode is satisfied. If this is the first iteration an automatic "yes" results; for other iterations, if a "no" results, the step size for incrementing the compensator coefficients is halved and the iteration is repeated. This process continues until the mode is satisfied or until the step size becomes less than the minimum step size specified; in the latter case an automatic termination results (not shown in Fig. 3). The unsatisfied margins (active constraints) are determined and the partial vectors with respect to the compensator coefficients for each are calculated; this continues until partial vectors for all margins of all systems are computed.

The stop condition is checked next: if "yes," pertinent output is printed and a "STOP" results; if "no," the design specifications are checked. If these specifications are satisfied, again pertinent information is printed and a "STOP" results. Assuming a "no" results, information is printed only if this is a print iteration.

The computation of the directional change vector employs the constraint improvement technique (CIT). Provided the partial vectors are linearly independent and none are identically zero, the CIT produces a change vector that assures a TIFR and, consequently, an SIFR exists. If the CIT fails, a "STOP" condition is invoked. Otherwise, compensator poles and zeros on the constraint boundaries specified by the user are checked for directions of movements. If the directions of movements are such that violations of these zeta boundaries are inevitable, selected terms of the partial vectors are zeroed and the directional vector is recomputed. This method is continued until an inevitable violation will not occur.

The step size is then increased and the compensator coefficients are incremented according to the directional vector. After incrementation the compensators are checked for violations of the constraint boundaries. If violations occur, the step size is reduced and the compensator coefficients are reincremented; otherwise, a return to the beginning of the iterative process results and a new iteration begins.

## Acknowledgments

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